Non-Markovian Lévy diffusion in nonhomogeneous media

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We study the diffusion equation with a position-dependent, power-law diffusion coefficient. The equation possesses the Riesz-Weyl fractional operator and includes a memory kernel. It is solved in the diffusion limit of small wave numbers. Two kernels are considered in detail: the exponential kernel, for which the problem resolves itself to the telegrapher's equation, and the power-law one. The resulting distributions have the form of the Lévy process for any kernel. The renormalized fractional moment is introduced to compare different cases with respect to the diffusion properties of the system.

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I. INTRODUCTION

Diffusion processes are usually described in terms of either differential or fractional equations which contain a constant diffusion coefficient. In many physical problems, however, that coefficient depends on the position variable and such dependence is important [1]. As a typical example can serve the transport in porous, inhomogeneous media and in plasmas. Modeling the aggregation of interacting particles must take into account nonlocal effects, since the particle mobility depends on the average density [2]: the coalescence of particles results from long-range interactions (the Poisson-Smoluchowski paradigm) and the corresponding evolution equations contain a position-dependent coefficient. That modeling can be accomplished directly, via the nonlocal Fokker-Planck equation, in which the term with the space derivative is multiplied by a kernel and integrated over the position [3]. A similar method, applicable to the Lévy processes, consists in the integrating over the Lévy index, with some kernel (the distributed order space fractional equation) [4]. The spatial inhomogeneity can be also taken into account as an external potential which may substantially change the diffusive properties of the stochastic system, in particular of the Lévy flights [5].

The Lévy distributions constitute the most general class of stable processes and the Gaussian distribution is their special case. One can expect that the Lévy (and non-Gaussian) distributions emerge in transport processes for which the observable values experience long jumps—e.g., due to the existence of long-range correlations. The theory of Lévy flights is applicable to problems from various branches of science and technology. Moreover, the handling of specific and realistic systems often requires taking into account both memory effects and inhomogeneity of the media. As a typical example of the nonhomogeneous problem can serve the diffusion in the porous media, they often display fractal structure and the diffusion on the macroscale and mesoscale can be expressed by a stochastic equation driven by the Lévy process [6]. In general, the transport in fractal media can be described by the fractional Fokker-Planck equation with a variable, position-dependent, diffusion coefficient [7–9]. The Lévy flights bring about accelerated diffusion in the reaction-diffusion systems [10], and the probability distribution for that process is expressed by the fractional Fisher-Kolmogorov equation. The Lévy processes are typical for problems of high complexity, in particular in biological systems [11] where fractal structures are also encountered. For example, lipid diffusion in biomembranes has the characteristics of the Lévy process but it can no longer be regarded as Markovian. The theory of Nonnenmacher [12] takes into account memory effects, as well as the fractal structure of the medium; the diffusion coefficient depends on the variable diameter of the holes in the solvent through which the molecules jump. Application of the Lévy processes is natural also in many social and environmental problems. Recently, it has been established [13] that people mobility, estimated by bank note circulation and studied in terms of stochastic fractional equations, strongly depends on the geographical and sociological conditions. Therefore, its study requires including position-dependent quantities. That problem is directly related to the spread of infectious diseases. It has been demonstrated in the example of the SARS epidemic and by means of percolation model simulations [14] that the disease can spread very rapidly. Usually one assumes that the infection probability at a given distance is Lévy distributed due to long-range interactions but the process is local in time [15]. On the other hand, the percolation model of epidemics developed in Ref. [16] is restricted to short-range interactions (is local in space) but it introduces incubation times which obey Lévy statistics and then the model is non-Markovian.

In Refs. [17,18], the master equation for a jumping process, stationary and Markovian, has been studied. That process is a version of the coupled continuous-time random walk (CTRW), defined in terms of two probability distributions: the Poissonian waiting time distribution with position-dependent jumping frequency and a jump-size distribution. The standard technique to handle such master equations is the Kramers-Moyal expansion which produces the Fokker-Planck equation for the Gaussian jumping size distribution and it yields correct results for large times and large distances [19]. For the Lévy-distributed jumping size, the Fokker-Planck equation becomes the fractional diffusion equation, with the Riesz-Weyl fractional operator and the variable coefficient $D(x)$. Formally, it can be derived from the master equation by taking the Fourier transform and by neglecting higher terms in the wave number expansion of the jumping-size distribution (the diffusion approximation) [20]. The equation reads

$$\frac{\partial p(x,t)}{\partial t} = K^{\mu}\frac{\partial^{\mu}[D(x)p(x,t)]}{\partial|x|^{\mu}}, \tag{1}$$

where $1<\mu<2$. Since the diffusion coefficient is just the jumping frequency, the medium inhomogeneity enters the problem via the x-dependent waiting time distribution. For the Gaussian case ($\mu=2$), all kinds of diffusion, both normal and anomalous, are predicted [20] and they are determined by the jumping frequency.

Equation (1) can be regarded as a special case of a more general problem than the random walk and which traces back to the microscopic foundations of nonequilibrium statistical mechanics. The well-known achievement of Zwanzig [21] was the derivation of the non-Markovian kinetic equation for the probability distribution in the space of macroscopic-state variables. More precisely, by starting from the Liouville–von Neumann equation for the density operator ρ, $i\hbar\partial\rho/\partial t=[H,\rho]$, where H is the Hamiltonian of the system, one can obtain the generalized master equation

$$\frac{\partial P_{\xi}(t)}{\partial t} = \int_0^t dt'\,\phi(t-t')\sum_{\mu}[F_{\xi\mu}P_{\mu}(t') - F_{\mu\xi}P_{\xi}(t')], \tag{2}$$

where P_{ξ} denotes the diagonal elements of the density matrix and $F_{\mu\xi}$ are the transition rates [22]. Then the equation is non-Markovian and it contains the memory kernel $\phi(t)$. Markovian equations like Eq. (1) follow from the generalized master equation if memory effects are negligible. However, usually this is not the case. We have already discussed examples of Lévy processes, with power-law tails of the distribution, which exhibit memory effects. In fact, the importance of these effects was realized a long time ago—e.g., in the description of the resonance transfer of the excitation energy between molecules [22]. The detailed calculation for the anthracene molecules shows that the memory kernel is exponential and the generalized, nonlocal-in-time, master equation must be applied to get proper results for small times. One can expect that memory effects are still more pronounced for systems with the characteristic decay rate slower than exponential, which often happens for atomic and molecular systems [23]. Stochastic dynamical processes are generally nonlocal in time due to the finite time of the interaction with the environment. Moreover, for a stochastic system which is coupled to a fractal heat bath via a random matrix interaction [24], finite correlations emerge and its relaxation has to be described in terms of the generalized, non-Markovian Langevin equation, with the memory friction term [25,26]. Also the speed of transport is affected by the memory. In the non-Markovian CTRW processes it hampers the dynamics and such systems are subdiffusive [27]. Such processes are described by the generalized master equation, with a memory kernel, if the waiting time distribution possesses long, algebraic tails. That equation follows directly from the generalized Chapman-Kolmogorov equation which determines the probability distribution in the phase space [28,29].

By applying the nearest-neighbor approximation to the transition rates $F_{\mu\xi}$ and taking the continuum limit [22], one obtains from Eq. (2) the non-Markovian Fokker-Planck equation. In the presence of long-range correlations, however, the nearest-neighbor approximation is no longer valid. If the transition rates are symmetric and distributed according to Lévy statistics in the continuum limit, the Kramers-Moyal expansion produces a fractional derivative, instead of a Gaussian. Then, for the variable diffusion coefficient $D(x)$, the equation which corresponds to Eq. (2) becomes

$$\frac{\partial p_{\gamma}(x,t)}{\partial t} = \int_0^t K_{\gamma}(t-t')\mathcal{L}_x[p_{\gamma}(x,t')]dt', \tag{3}$$

where the operator

$$\mathcal{L}_x = K^{\mu}\frac{\partial^{\mu}}{\partial|x|^{\mu}}D(x) \tag{4}$$

acts only on the x variable. The parameter γ measures the rate of memory loss.

Equation (3) is of interest both from quantum and classical point of view. In the atomic and molecular physics—e.g., in a few-mode spin boson model [30] and random-matrix theory [31]—where the decay rate is slow, an equation analogous to Eq. (3) can be applied. The operator $\mathcal{L}_x$ is then expressed in terms of the "superoperator" which represents an instantaneous intervention of the environment over the system [23] and it can assume a quite general form. In the classical context, Eq. (3) has been discussed in Ref. [32]; the operator $\mathcal{L}_x$ has the Fokker-Planck form in this case, with a constant diffusion coefficient and a potential force.

In this paper we study the diffusion problem for systems which are driven by the Lévy-distributed transition rate and for which both medium inhomogeneity and memory effects are important. We assume $D(x)=|x|^{-\theta}(\theta>-1)$. The power-law form of the diffusion coefficient has been used to describe some physical phenomena—e.g., the transport of fast electrons in a hot plasma [33] and turbulent two-particle diffusion [34]. It is also used in theoretical analyses of the fractional Fokker-Planck equation [35–38]—e.g., as an ansatz for the problem of diffusion in fractal media [7–9,39]. Obviously, for the Markovian case $K_{\gamma}(t)=\delta(t)$, Eq. (3) resolves itself to Eq. (1).

In Sec. II we solve the fractional telegrapher's equation which follows from Eq. (3) for the case of the exponential memory kernel $K_{\gamma}(t)$. The solution for an arbitrary kernel, expressed in the form of the Laplace transform, is derived in Sec. III. Moreover, the case of the power-law kernel is solved in details. In Sec. IV we derive the fractional moments and discuss their application to a description of the diffusion process. The results presented in the paper are summarized in Sec. V.

II. EXPONENTIAL KERNEL

If the memory effects are weak, we can assume that the kernel $K_{\gamma}(t)$ decays exponentially. Then let us consider the following kernel:

$$K_{\gamma}(t) = \gamma e^{-\gamma t} \quad (\gamma>0), \tag{5}$$

which becomes the δ function in the limit $\gamma\to\infty$ (the Markovian case). In this case, the integral equation (3) reduces itself to a differential equation. It can be derived by inserting Eq. (5) into Eq. (3) and by differentiating twice over time, in order to get rid of the integral. Finally, we get the equation

$$\frac{\partial^2 p_\gamma(x,t)}{\partial t^2} + \gamma \frac{\partial p_\gamma(x,t)}{\partial t} = K^\mu \gamma \frac{\partial^\mu [|x|^{-\theta} p_\gamma(x,t)]}{\partial |x|^\mu}, \tag{6}$$

which is a generalized and fractional version of the well-known telegrapher's equation. Originally, the telegrapher's equation, which is the hyperbolic one, was introduced into the theory of stochastic processes by Cattaneo [40] in order to avoid infinitely fast propagation for very small times. Its fractional generalization describes, e.g., a two-state process with correlated noise [41] and it predicts an inhanced diffusion in the limit of long time. On the other hand, in the case of the divergent second moment, the telegrapher's equation with Riesz-Weyl derivative results from the fractional Klein-Kramers equation for the Lévy-distributed jumping size [29]. In that equation, the parameter γ has the sense of a damping constant in the corresponding Langevin equation.

In the diffusion limit of small wave numbers, the Markovian equation (1) is satisfied by the Fox function $H_{2,2}^{1,1}$ [20]. Since our main objective is to study the diffusion problem, we restrict also the present analysis to that limit. We will try to find the solution of Eq. (6) in the same form as for the Markovian equation. Therefore we assume

$$p_\gamma(x,t) = NaH_{2,2}^{1,1}\left[a|x| \,\middle|\, \begin{matrix} (a_1,A_1),(a_2,A_2) \\ (b_1,B_1),(b_2,B_2) \end{matrix} \right], \tag{7}$$

where the time dependence is restricted to the function $a(t)$ and N is the normalization constant. The method of solution, described in Ref. [20], consists in the insertion of the Fourier transform of expression (7) into the Fourier-transformed equation (6). Then we determine the coefficients of the Fox function by demanding that Eq. (6) be satisfied in the diffusion limit—i.e., for small wave numbers. In fact, the latest assumption does not introduce any additional idealization since the equation itself is valid only in the diffusion limit.

We start with the Fourier transform of Eq. (6); it reads

$$\frac{\partial^2 \tilde{p}_\gamma(k,t)}{\partial t^2} + \gamma \frac{\partial \tilde{p}_\gamma(k,t)}{\partial t} = -K^\mu \gamma |k|^\mu \mathcal{F}[|x|^{-\theta} p_\gamma(x,t)]. \tag{8}$$

The Fourier transform of the Fox function can be expressed also in terms of the Fox function (for the definition and some useful properties of the Fox functions see Ref. [20] and references therein). Due to the multiplication rule, the product $|x|^{-\theta}p_\gamma(x,t)$ is the Fox function as well. Both sides of Eq. (8) can now be expanded in series of fractional powers of $|k|$. We can satisfy Eq. (8) by a suitable choice of parameters of the function (7) and by neglecting terms higher than $|k|^\mu$. The results are the following. The expansion of the functions on the left-hand side (LHS) and RHS, respectively, reads $\tilde{p}_\gamma(k,t) \approx 1 - Nh_\mu a^{-\mu}|k|^\mu$ and $\mathcal{F}[|x|^{-\theta}p_\gamma(x,t)] \approx Nh_0^{(\theta)}a^\theta$, with the following coefficients: $h_0^{(\theta)} = 2(\mu+\theta)/(2+\theta)$ and $h_\mu = -2\frac{(\mu+\theta)^2}{\pi}\Gamma(-\mu)\Gamma(\mu+\theta)\cos(\mu\pi/2)\sin(\frac{\mu+\theta}{2+\theta}\pi)$. The vanishing of all other terms of order less than μ is the necessary condition to satisfy Eq. (8). The solution takes the form

$$p_\gamma(x,t) = NaH_{2,2}^{1,1}\left[a|x| \,\middle|\, \begin{matrix} \left(1 - \frac{1-\theta}{\mu+\theta}, \frac{1}{\mu+\theta}\right), \left(1 - \frac{1-\theta}{2+\theta}, \frac{1}{2+\theta}\right) \\ (\theta,1), \left(1 - \frac{1-\theta}{2+\theta}, \frac{1}{2+\theta}\right) \end{matrix} \right], \tag{9}$$

and the coefficients b_1 and B_1 are responsible for the distribution behavior near $x=0$. b_1 and B_1 cannot be determined in the diffusive limit, and they are meaningless from the point of view of the diffusion process; the values θ and 1 we have assumed correspond to the jumping process, considered in Ref. [20]. Generally, Eq. (3) is satisfied by the function (9) for any choice of the coefficients b_1 and $B_1 > 0$, such that $b_1 \to 0$ and $B_1 \to 1$ for $\theta \to 0$. The normalization factor N can be determined in a simple way from the formula $N = [2\chi(-1)]^{-1}$, where $\chi(-s)$ is the Mellin transform of the Fox function. A simple algebra yields

$$N = -\frac{\pi}{2}\left[\Gamma(1+\theta)\Gamma\left(-\frac{\theta}{\mu+\theta}\right)\sin\left(\frac{\theta}{2+\theta}\pi\right)\right]^{-1}. \tag{10}$$

Alternatively, since $|k|$ is small, we can express the Fourier transform of the solution as

$$\tilde{p}_\gamma(k,t) \approx 1 - \sigma^\mu |k|^\mu \approx \exp(-\sigma^\mu |k|^\mu), \tag{11}$$

where

$$\sigma^\mu = K^{-\mu} \frac{(\mu+\theta)^2\Gamma(-\mu)\Gamma(\mu+\theta)\cos(\mu\pi/2)\sin\left(\frac{\mu+\theta}{2+\theta}\right)}{\Gamma(1+\theta)\Gamma\left(-\frac{\theta}{\mu+\theta}\right)\sin\left(\frac{\theta}{2+\theta}\pi\right)} a^{-\mu}. \tag{12}$$

Equation (11) means that the solution of Eq. (3) coincides with the Lévy process in the limit $k \to 0$. Then the solution

(9) can be expressed in the form which is generic for any symmetric Lévy distribution [42]:

$$p_\gamma(x,t) = \frac{1}{\mu\sigma} H_{2,2}^{1,1}\left[\frac{|x|}{\sigma} \left| \begin{matrix} (1-1/\mu, 1/\mu), (1/2,1/2) \\ (0,1), (1/2,1/2) \end{matrix} \right. \right]. \tag{13}$$

Formula (12) establishes the relation between the solutions (9) and (13). Those expressions are equivalent only in the limit $k \to 0$, and they behave differently for small $|x|$. We will demonstrate in Sec. III that the Lévy process is the solution of Eq. (3) for any kernel. Therefore, the form (13) is quite universal and we apply it in the following. The problem is reduced in this way to evaluating the function $\sigma(t)$.

Now, Eq. (8) becomes the ordinary differential equation

$$\frac{1}{\gamma}\ddot{\xi} = -\dot{\xi} + K^\mu \frac{h_0^{(\theta)}}{h_\mu}\xi^{-\theta/\mu}, \tag{14}$$

where $\xi(t) = a^{-\mu}$. We assume the following initial conditions: $\xi(0) = \dot{\xi}(0) = 0$ which correspond to the condition $p_\gamma(x,0) = \delta(x)$. Equation (14) has the structure of the equation of motion with a "friction term," a positive "driving force," and a "mass" $1/\gamma$. The meaning of the quantity ξ, the time evolution of which Eq. (14) describes, remains to be determined. The variable ξ, as well as $\dot{\xi}$, rises with time and finally the balance of "forces," given by the equation

$$\dot{\xi} - K^\mu \frac{h_0^{(\theta)}}{h_\mu}\xi^{-\theta/\mu} = 0, \tag{15}$$

is reached. Note that the above expression is equivalent to Eq. (14) in the Markovian limit $\gamma \to \infty$. Therefore $p_\gamma(x,t) = p_\infty(x,t) = p(x,t)$ for $t \to \infty$. The solution of Eq. (15) produces the result

$$a(t) = \left[K^\mu \frac{h_0^{(\theta)}}{h_\mu}\left(1 + \frac{\theta}{\mu}\right)t \right]^{-1/(\mu+\theta)} \quad (t \to \infty), \tag{16}$$

which corresponds to the exact solution (for arbitrary time) for the Markovian limit, $p(x,t)$.

The case of the constant diffusion coefficient, $\theta=0$, is a particular case, and it can be solved easily. Solution of Eq. (14) leads to the result

$$a(t) = \frac{1}{K}\left[-\frac{1}{\gamma}(1 - e^{-\gamma t}) + t \right]^{-1/\mu}. \tag{17}$$

For $\theta \neq 0$ and arbitrary time, Eq. (14) can be solved by numerical integration and the distribution $p_\gamma(x,t)$ determined from Eq. (13). To evaluate the Fox function we use the general formula for its series expansion and then Eq. (13) can be expressed in the computable form

$$p_\gamma(x,t) = \frac{1}{\pi\sigma\mu}\sum_{n=0}^{\infty} \frac{\Gamma[1 + (2n+1)/\mu]}{(2n+1)!!}(-1)^n \left(\frac{x}{\sigma}\right)^{2n}. \tag{18}$$

Figure 1 presents some exemplary probability distributions, so chosen to illustrate the influence of memory on the time evolution. Since the series (18) is poorly convergent, evaluation of the distribution for large $|x|$ required quadruple computer precision [43]. The case with $\gamma=3$ is close to the Markovian one; a comparison with the case characterized by long memory shows that the spread of the distribution slows down with a decreasing value of γ—i.e., for stronger memory [larger "inertia" in Eq. (14)]. In the limit $t \to \infty$ the curves which correspond to different γ values and the same θ coincide.

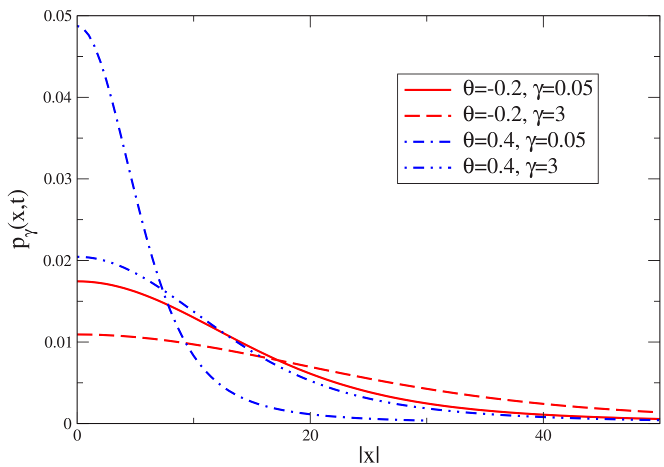

FIG. 1. (Color online) Probability distributions for the case of the exponential kernel with $\mu=1.5$, calculated from Eqs. (14), (12), and (18), for $t=50$. The initial condition is $p_\gamma(x,0) = \delta(x)$.

III. GENERAL CASE

The description by means of Eq. (3) with exponential memory kernel does not apply to systems with long-time correlations and small decay rate. In the study of realistic systems one encounters a variety of forms of the kernel; some of them are very complex. It is typical for natural signals that they do not represent a simple kinetics, characterized by a unique Hurst exponent. Random processes which take into account the whole spectrum of the time-dependent Hurst exponents serve then as useful models. This concept, applied to the fractional equation formalism, leads to integration over the order of differentiation (the distributed-order diffusion equation) [4] and the kernel assumes the integral form $\int f(\alpha)t^{-\alpha}\, d\alpha$. Reactions in polymer systems are also described by using complicated kernels [44,45]. Therefore, in this section we consider Eq. (3) for the case as general as possible. We will demonstrate that the solution in a closed form can be obtained for the arbitrary kernel.

Equation (3) has the structure of the Volterra integro-differential equation with a kernel which depends on the difference of its arguments. Therefore, methods using Laplace transforms are applicable. Following Sokolov [32], we apply a method of the integral decomposition which allows us to express the required solution by solution of the corresponding Markovian equation (1). Let us define the function $T(\tau,t)$ by its Laplace transform

$$T^\star(\tau,u) = \frac{1}{K_\gamma^\star}\exp\left(-\tau\frac{u}{K_\gamma^\star}\right). \tag{19}$$

If we know the function $T(\tau,t)$, the probability distribution $p_\gamma(x,t)$ can be obtained by a simple integration:

$$p_\gamma(x,t) = \int_0^\infty p(x,\tau)T(\tau,t)d\tau. \tag{20}$$

However, since inversion of Eq. (19) is difficult for any kernel, it is expedient to get rid of T. To achieve that, we take the Fourier transform from Eq. (20) and eliminate the function $T(\tau,t)$ by using the definition (19). The final solution is of the form of the following Fourier-Laplace transform:

$$\tilde{p}_\gamma^\star(k,u) = \frac{1}{K_\gamma^\star}\tilde{p}^\star\left(k,\frac{u}{K_\gamma^\star}\right). \tag{21}$$

The above formalism can be applied for any kernel K_γ and any operator $\mathcal{L}_x$; the main difficulty consists in inversion of the Laplace transforms.

First of all, we find that if the Markovian process $p(x,t)$ is Lévy distributed in the diffusion limit $k\to 0$, then the non-Markovian process is also the Lévy process in this limit. Indeed, the Fourier transform of the Markovian solution is given by Eq. (11), where $\sigma(t)$ follows from Eqs. (12) and (16). Then we take the Laplace transform from that expression and insert the result into Eq. (21): $\tilde{p}_\gamma^\star(k,u)=1/u-F^\star(u)|k|^\mu$. Finally, the inversion of the Laplace transform yields

$$\tilde{p}_\gamma(k,t) = 1 - F(t)|k|^\mu, \tag{22}$$

which is just the Fourier representation of the Lévy distribution for small $|k|$. To get the function $F(t)$ we need to invert the Laplace transform

$$F^\star(u) = \frac{Nh_\mu}{K_\gamma^\star(u)}[a^{-\mu}]^\star(u/K_\gamma^\star) \tag{23}$$

and we assume that this inverse transform exists. The solution is given by Eq. (13), where $\sigma(t)=[F(t)]^{1/\mu}$.

We will consider two particular cases in detail. In the case of the exponential memory kernel (5), discussed already in Sec. II, we have $K_\gamma^\star(u)=\gamma/(u+\gamma)$ and Eq. (21) produces the result

$$\tilde{p}_\gamma^\star(k,u) = \frac{1}{u} - a_0\Gamma(1+\alpha)\gamma^\alpha|k|^\mu u^{-(\alpha+1)}(u+\gamma)^{-\alpha}, \tag{24}$$

where $a_0=Nh_\mu^{1-\alpha}(K^\mu h_0^{(\theta)}\alpha/\mu)^\alpha$ and $\alpha=\mu/(\mu+\theta)$. The above expression cannot be inverted in closed form. However, if we are interested only in large times, the last term in Eq. (24) can be expanded around $u=0$: $(u+\gamma)^{-\alpha}\approx\gamma^{-\alpha}-\alpha\gamma^{-(\alpha+1)}u$. Then inversion of the Laplace transform yields

$$\tilde{p}_\gamma(k,t) = 1 - a_0|k|^\mu\left(t^\alpha - \frac{\alpha^2}{\gamma}t^{\alpha-1}\right) \quad (t\to\infty) \tag{25}$$

and this expression demonstrates how the solution p_γ approaches its asymptotic, Markovian form. The final solution, valid for large t, is given by Eq. (13) with $\sigma=\left[a_0\left(t^\alpha-\frac{\alpha^2}{\gamma}t^{\alpha-1}\right)\right]^{1/\mu}$.

The other physically important kernel has the power-law form, with long tails,

$$K_\gamma(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \quad (0<\gamma<1). \tag{26}$$

Equation (3) with the kernel (26) is usually presented as a fractional equation with Riemann-Liouville derivative [46]—which is equivalent to the Caputo operator for a special choice of the initial conditions—in the form

$$\frac{\partial p_\gamma(x,t)}{\partial t} = {}_0D_t^{\gamma-1}\mathcal{L}_x[p_\gamma(x,t)]. \tag{27}$$

The power-law kernels are used to describe subdiffusive relaxation, e.g., in the framework of the CTRW [27]. They emerge also as a result of the coupling to the fractal heat bath via the random matrix interaction [24]. To solve Eq. (3) we follow the same procedure as for the exponential kernel. The Laplace transform of Eq. (26) reads $K_\gamma^\star(u)=u^{\gamma-1}$, and Eq. (21) takes the form

$$\tilde{p}_\gamma^\star(k,u) = \frac{1}{u} - a_0\Gamma(1+\alpha)\gamma^\alpha|k|^\mu u^{-2\alpha+\gamma\alpha-1}. \tag{28}$$

The inversion can be easily performed:

$$\tilde{p}_\gamma(k,t) = 1 - \frac{a_0\gamma^\alpha\Gamma(1+\alpha)}{\Gamma(2\alpha-\gamma\alpha+1)}|k|^\mu t^{2\alpha-\gamma\alpha} \equiv 1 - F(t)|k|^\mu. \tag{29}$$

Clearly, the above solution does not converge with time to the Markovian asymptotics, $F(t)\sim t^\alpha$, in contrast to the case of the exponential kernel.

To conclude this section, we want to mention yet another approach to Eq. (3), which is a direct generalization of the method applied for the telegrapher's equation in Sec. II. Inserting the expansion of the functions $\tilde{p}_\gamma(k,t)$ and $\mathcal{F}[|x|^{-\theta}p_\gamma(x,t)]$ into Eq. (3) confirms the finding that the solution can be expressed in terms of the Fox function $H_{2,2}^{1,1}$ and it is Lévy distributed. The resulting equation is a generalization of Eq. (14), and it determines the function $\sigma(t)$:

$$\frac{d\xi}{dt} = K^\mu\frac{h_0^{(\theta)}}{h_\mu}\int_0^t K_\gamma(t-t')\xi^{-\theta/\mu}\,dt'. \tag{30}$$

Mathematically, Eq. (30) has the form of the nonlinear Volterra integro-differential equation. Since the numerical inversion of the Laplace transforms is not always an easy task (methods are often unstable), numerical solution of Eq. (30) could be a useful alternative to Eq. (22).

IV. DIFFUSION

The diffusion process is usually characterized by the time dependence of the second moment of the probability distribution: if this dependence is linear in the limit of long time, the diffusion is called normal. There are many examples of physical systems for which the variance rises faster than linearly with time (hyperdiffusion) or slower (subdiffusion). Such behaviors are typical for transport in disordered media [47] and systems with traps and barriers. In the realm of dynamical systems, a substantial acceleration of the diffusion is caused by the presence of regular structures in the phase space [48]. On the other hand, the subdiffusion appears in the non-Markovian version of CTRW, as a result of a non-Poissonian, power-law form of the waiting time distribution [27].

When we enter the field of Lévy processes, the situation becomes more complicated. The stochastic variable performs very long jumps and their size is limited only by the size of whole system. As a result, the second moment, as well as all moments of the order μ or higher, is divergent. Mathematically, that follows from the fact that the tail of the Lévy distribution is the power law $\sim|x|^{-(1+\mu)}$. Therefore, one cannot describe the diffusion process in terms of the position variance and some other quantity which could serve as an estimation of the speed of transport is needed. One possibility is to consider still the second moment but with integration limits restricted to a time-dependent interval (the walker in the imaginary growing box) [49]. On the other hand, one can derive fractional moments of order $\delta<\mu$.

By derivation of the moments of the probability distribution $p_\gamma(x,t)$, Eq. (13), we utilize simple properties of the Mellin transform from the Fox function

$$\begin{aligned}\langle|x|^\delta\rangle &= 2\int_0^\infty x^\delta p_\gamma(x,t)dx = \frac{2}{\mu}\sigma^\delta\chi(-\delta-1)\\ &= \frac{2}{\pi}\sigma^\delta\Gamma(\delta)\Gamma\left(1-\frac{\delta}{\mu}\right)\sin(\delta\pi/2). \qquad (31)\end{aligned}$$

Let us consider two quantities: the renormalized moment of order μ, defined by the expression

$$\mathcal{M}^\mu = \lim_{\epsilon\to 0^+}\epsilon\langle|x|^{\mu-\epsilon}\rangle = \frac{2}{\pi}\sigma^\mu\Gamma(\mu)\sin(\mu\pi/2), \qquad (32)$$

where we applied the property $\Gamma(x)\to 1/x$ for $x\to 0$, and then the fractional diffusion coefficient $\mathcal{D}^{(\mu)}(t)=\frac{1}{\Gamma(1+\mu)}\frac{1}{t}\mathcal{M}^\mu$. In the Markovian case, defined by Eq. (1), the coefficient $\mathcal{D}^{(\mu)}$ is useful to classify the diffusion: for $\theta<0$ it rises with time, for $\theta>0$ it falls, and it converges to a constant for $\theta=0$ [20]. That pattern is consistent with the diffusion properties, defined in the ordinary sense, of the Fokker-Planck equation ($\mu=2$). Therefore, in the following we will name all kinds of the diffusion—the subdiffusion, the normal diffusion, and the superdiffusion—according to the time dependence of the coefficient $\mathcal{D}^{(\mu)}$.

We begin with the case of the exponential kernel. First we realize that, since $\sigma^\mu=Nh_\mu\xi$, the renormalized moment $\mathcal{M}^\mu$ is directly related to the variable ξ: $\mathcal{M}^\mu=\frac{2}{\pi}Nh_\mu\Gamma(\mu)\sin(\mu\pi/2)\xi$. Therefore, interpretation of Eq. (14) is straightforward: it describes the deterministic time evolution of the moment $\mathcal{M}^\mu$. The diffusion properties of the system remain unchanged, compared to the Markovian case, because in the limit $t\to\infty$ both solutions coincide. However, at small time the influence of the memory, which hampers both the spread of the distribution and the relaxation to the Markovian asymptotics, is visible. Figure 2 illustrates that

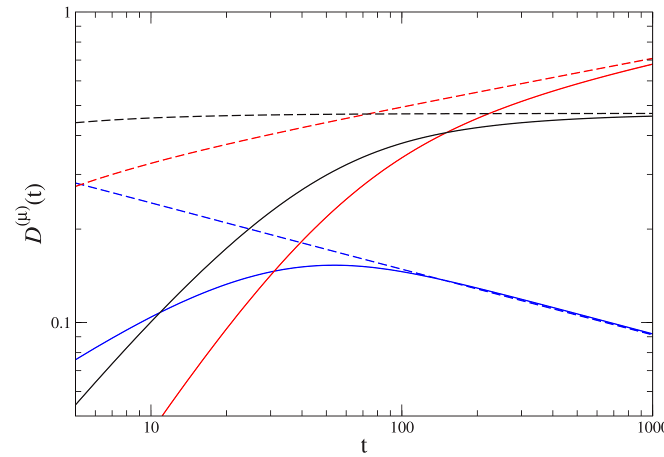

FIG. 2. (Color online) The fractional diffusion coefficient for the case of the exponential kernel with $\gamma=0.05$ (solid lines) and $\gamma=3$ (dashed lines), as a function of time, obtained from numerical solution of Eq. (14). Results for three values of θ are presented: $\theta=-0.2$ (upper lines for large t), $\theta=0$ (middle lines), and $\theta=0.4$ (lower lines); $\mu=1.5$.

effect for three values of θ which have different sign. The asymptotic, Markovian limit is achieved first for the subdiffusive case $\theta=0.4$.

For the case of the power-law kernel we calculate the fractional diffusion coefficient by means of Eq. (32); the quantity $\sigma^\mu(t)=F(t)$ is given by Eq. (29). We obtain

$$\begin{aligned}\mathcal{D}^{(\mu)}(t) &= \frac{2a_0\gamma^\alpha\Gamma(1+\alpha)}{\pi\mu\Gamma(\alpha-\gamma\alpha+1)}\sin(\mu\pi/2)t^{2\alpha-\gamma\alpha-1}\\ &\sim t^{[\mu(1-\gamma)-\theta]/(\mu+\theta)}. \qquad (33)\end{aligned}$$

The diffusion properties of the system follow directly from the above formula. The influence of the parameter θ, which quantifies the structure of the medium, is similar as in the Markovian case [20]: the larger θ is, the weaker is the diffusion. For $\theta\leqslant 0$, there is clearly superdiffusion. For positive θ, the diffusion becomes weaker with θ and finally it turns into subdiffusion; the critical value, which corresponds to the normal diffusion, is $\theta_{cr}=\mu(1-\gamma)$. On the other hand, if $0<\theta<\mu$, there is a critical value of γ which separates the superdiffusion from the subdiffusion: $\gamma_{cr}=1-\theta/\mu$. For $\theta>\mu$ the motion is always subdiffusive. The parameter γ measures the degree of time nonlocality; it is the largest if γ approaches 0. The diffusion speed grows if γ diminishes because the system behavior at large times becomes sensitive to the initial stages of the evolution when the distribution spreads rapidly. The latter conclusion shows that memory can influence the diffusion in many ways: the non-Markovian CTRW predicts the weakening of the diffusion, and it is just a consequence of memory in the system [27]. However, in that case time nonlocality invokes a trapping mechanism.

Note that the above properties, in particular the presence of a transition from the subdiffusion to the superdiffusion when changing the parameters of the system, still hold if one considers some other fractional moment of order $\delta<\mu$, instead of the renormalized moment $\mathcal{M}_\mu$.

For any kernel K_γ, except for the δ function and for the exponential kernel, the time evolution of the moment $\mathcal{M}_\mu$ is governed by the nonlocal equation (30) and the diffusion properties follow from its solution. In fact, looking for the full solution may be avoided in some cases: the kind of diffusion is already determined by the sign of the function $\ddot{\xi}(t)$ in the limit of long time.

V. SUMMARY AND DISCUSSION

We have studied the diffusion process for non-Markovian systems with position-dependent diffusion coefficient, which involves Lévy flights, and then the variance of the corresponding probability distribution is infinite. The integral equation for that problem contains the fractional Riesz-Weyl operator and the time-dependent memory kernel; the diffusion coefficient depends on the position in an algebraic, scaling way. The equation has been solved in terms of the Fox functions in the limit of small wave numbers. We have demonstrated that this solution represents the Lévy process for any memory kernel. The formal solution has been obtained in closed form which involves the Laplace transform. The inversion of that transform may be a difficult task for most of the kernels and then numerical methods have to be applied. Two forms of the kernel have been discussed in detail: the exponential kernel, for which the problem resolves itself to the generalized telegrapher's equation, and power-law one, which is equivalent to the fractional equation with both the Riesz-Weyl operator and the Riemann-Liouville fractional operator. For the exponential kernel, memory initially slows down the spread of the distribution but asymptotically the solution converges to that of the Markovian equation. The case with a power-law kernel reveals much more interesting behavior. There is an interplay among all ingredients of the dynamics, in particular between the range of the memory γ and the inhomogeneity parameter θ, which can result in all kinds of diffusion, both normal and anomalous. In order to make that classification possible, we have introduced the fractional diffusion coefficient, defined in terms of the renormalized moment of order μ. This coefficient allows us to maintain the standard terminology of the anomalous diffusion also for the Lévy flights.

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